

On One-Dimensional Quaternion Fourier Transform

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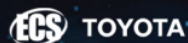
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On One-Dimensional Quaternion Fourier Transform

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Abstract. There have been several efforts in the literature to extend the traditional Fourier transformation by using the quaternion algebra. This paper presents the one-dimensional quaternion Fourier transform. We derive its properties which are the extensions of corresponding properties of the one-dimensional Fourier transformation. Finally, the convolution theorem related to the one-dimensional quaternion Fourier transform is discussed.

1. Introduction

As we know, the traditional Fourier transformation has been widely used in engineering, mathematics, statistics and computer sciences (see, e.g. [1, 2, 3, 4]). This fact motivates researchers to generalize the traditional Fourier transformation in various directions of transformations. For instance, in [5, 6, 7] the authors proposed the linear canonical transform, which is generalizations of the traditional Fourier transformation in the linear canonical domain. In [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], the authors introduced two-dimensional quaternion Fourier transform, which can be regarded as the extension of the two-dimensional Fourier transformation using the quaternions. Therefore, it is important to study the one-dimensional quaternion Fourier transform, which is generalizations of the traditional Fourier transformation in the framework of Hamiltonian quaternion algebra.

In this article, we introduce one-dimensional quaternion Fourier transform. We investigate its properties which are the extensions of the one-dimensional Fourier transformation. We finally develop the general convolution theorem related the one-dimensional quaternion Fourier transformation.

2. Notation

We begin to state some basic facts on quaternions. The quaternion algebra over real number \mathbb{R} is expressed as

$$\mathbb{H} = \{h = h_0 + \mathbf{i}h_1 + \mathbf{j}h_2 + \mathbf{k}h_3 \mid h_0, h_1, h_2, h_3 \in \mathbb{R}\}, \quad (1)$$

which is an associative four-dimensional algebra and satisfies:

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \quad (2)$$

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From (2) we see that quaternions can be considered as a noncommutative extension of complex numbers. The conjugate of a quaternion h is given by

$$\bar{h} = h_0 - \mathbf{i}h_1 - \mathbf{j}h_2 - \mathbf{k}h_3, \quad h_0, h_1, h_2, h_3 \in \mathbb{R}. \quad (3)$$

It satisfies the property as complex number property, i.e.,

$$\overline{hp} = \bar{p}\bar{h}. \quad (4)$$

In view of (3), we get the norm of h as

$$|h| = \sqrt{h\bar{h}} = \sqrt{h_0^2 + h_1^2 + h_2^2 + h_3^2}. \quad (5)$$

For every quaternion p, h , it holds

$$|hp| = |h||p| \quad (6)$$

The inverse of nonzero quaternion h is given by

$$h^{-1} = \frac{\bar{h}}{|h|^2}. \quad (7)$$

3. 1-D QFT and its properties

Let us present the definition of the one-dimensional quaternion Fourier transform (compare to [19]). We then explore several its properties.

Definition 3.1. The one-dimensional quaternion Fourier transform (qFT) of $g \in L^1(\mathbb{R}; \mathbb{H})$ is given by

$$\begin{aligned} \mathcal{F}_I\{g\}(\omega) &= \int_{\mathbb{R}} g(y) e^{-\mathbf{j}2\pi\omega y} dy \\ &= \int_{\mathbb{R}} (g_0(y) + \mathbf{i}g_1(y) + \mathbf{j}g_2(y) + \mathbf{k}g_3(y)) e^{-\mathbf{j}2\pi\omega y} dy \\ &= \mathcal{F}_I\{g_0\}(\omega) + \mathbf{i}\mathcal{F}_I\{g_1\} + \mathbf{j}\mathcal{F}_I\{g_2\} + \mathbf{k}\mathcal{F}_I\{g_3\}. \end{aligned} \quad (8)$$

From equation (8) above we will find that if $g(y)$ is a real-valued function, then we may interchange the position of the kernel $e^{-\mathbf{j}2\pi\omega y}$ as

$$\mathcal{F}_I\{g\}(\omega) = \int_{\mathbb{R}} g(y) e^{-\mathbf{j}2\pi\omega y} dy = \int_{\mathbb{R}} e^{-\mathbf{j}2\pi\omega y} g(y) dy. \quad (9)$$

Definition 3.2. The inverse can be expressed in the form

$$g(y) = \mathcal{F}_I^{-1}[\mathcal{F}_I\{g\}](y) = \int_{\mathbb{R}} \mathcal{F}_I\{g\}(\omega) e^{\mathbf{j}2\pi\omega y} d\omega. \quad (10)$$

Lemma 3.1. Suppose that $g(y)$ is a continuous differential function. If $g(y)$ belongs $L^1(\mathbb{R}; \mathbb{H})$, then

$$\mathcal{F}_I\left\{\frac{d^n g}{dx^n}\right\}(\omega) = \mathcal{F}_I\{g\}(\omega)(\mathbf{j}2\pi\omega)^n, \quad n \in \mathbb{N}. \quad (11)$$

Proof. In fact, we have for $n=1$,

$$\begin{aligned}\mathcal{F}_I\left\{\frac{dg}{dx}\right\}(\omega) &= \mathcal{F}_I\left\{\frac{dg_0}{dx}\right\}(\omega) + \mathbf{i}\mathcal{F}_I\left\{\frac{dg_1}{dx}\right\} + \mathbf{j}\mathcal{F}_I\left\{\frac{dg_2}{dx}\right\} + \mathbf{k}\mathcal{F}_I\left\{\frac{dg_3}{dx}\right\} \\ &= \mathcal{F}_I\{g_0\}(\omega)\mathbf{j}2\pi\omega + \mathbf{i}\mathcal{F}_I\{g_1\}(\omega)\mathbf{j}2\pi\omega + \mathbf{j}\mathcal{F}_I\{g_2\}(\omega)\mathbf{j}2\pi\omega + \mathbf{k}\mathcal{F}_I\{g_3\}(\omega)\mathbf{j}2\pi\omega \\ &= (\mathcal{F}_I\{g_0\}(\omega) + \mathbf{i}\mathcal{F}_I\{g_1\}(\omega) + \mathbf{j}\mathcal{F}_I\{g_2\}(\omega) + \mathbf{k}\mathcal{F}_I\{g_3\}(\omega))\mathbf{j}2\pi\omega \\ &= \mathcal{F}_I\{g\}(\omega)\mathbf{j}2\pi\omega.\end{aligned}\tag{12}$$

For general derivatives the theorem follows from mathematical induction. The proof is complete. \square

Theorem 3.2. For any $g, h \in L^1(\mathbb{R}; \mathbb{H})$, then one can get

$$\int_{\mathbb{R}} \mathcal{F}_I\{g\}(\omega) \overline{\mathcal{F}_I\{h\}(\omega)} d\omega = \int_{\mathbb{R}} g(y) \overline{h(y)} dy.\tag{13}$$

Proof. By using the definition qFT, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{F}_I\{g\}(\omega) \overline{\mathcal{F}_I\{h\}(\omega)} d\omega &= \int_{\mathbb{R}} \mathcal{F}_I\{g\} \int_{\mathbb{R}} \overline{h(y) e^{-\mathbf{j}2\pi\omega y}} dy d\omega \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{F}_I\{g\} e^{\mathbf{j}2\pi\omega y} d\omega \right) \overline{h(y)} dy \\ &= \int_{\mathbb{R}} g(y) \overline{h(y)} dy.\end{aligned}\tag{14}$$

The assertion is proved. \square

Likewise, we may obtain the following property.

Theorem 3.3. Under the assumptions of Theorem 3.2, one has

$$\begin{aligned}\int_{\mathbb{R}} g(\xi) \mathcal{F}_I\{h\}(\xi) d\xi \\ = \int_{\mathbb{R}} \mathcal{F}_I\{g\}(y) (h_0(y) + \mathbf{j}h_2(y)) dy + \int_{\mathbb{R}} \mathcal{F}_I\{g\}(-y) (\mathbf{i}h_1(y) + \mathbf{k}h_3(y)) dy.\end{aligned}\tag{15}$$

Proof. A first straightforward calculation shows that

$$\begin{aligned}\int_{\mathbb{R}} g(\xi) \mathcal{F}_I\{h\}(\xi) d\xi \\ = \int_{\mathbb{R}} g(\xi) \int_{\mathbb{R}} h(y) e^{-\mathbf{j}2\pi\omega y} dy d\xi \\ = \int_{\mathbb{R}} \int_{\mathbb{R}} g(\xi) (h_0(y) + \mathbf{i}h_1(y) + \mathbf{j}h_2(y) + \mathbf{k}h_3(y)) e^{-\mathbf{j}2\pi\omega y} dy d\xi \\ = \int_{\mathbb{R}} \int_{\mathbb{R}} g(\xi) (h_0(y) + \mathbf{j}h_2(y)) e^{-\mathbf{j}2\pi\omega y} dy d\xi + \int_{\mathbb{R}} \int_{\mathbb{R}} g(\xi) (\mathbf{i}h_1(y) + \mathbf{k}h_3(y)) e^{-\mathbf{j}2\pi\omega y} dy d\xi \\ = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} g(\xi) e^{-\mathbf{j}2\pi\omega y} d\xi \right] (h_0(y) + \mathbf{j}h_2(y)) dy + \int_{\mathbb{R}} \left[\int_{\mathbb{R}} g(\xi) e^{\mathbf{j}2\pi\omega y} d\xi \right] (\mathbf{i}h_1(y) + \mathbf{k}h_3(y)) dy \\ = \int_{\mathbb{R}} \mathcal{F}_I\{g\}(y) (h_0(y) + \mathbf{j}h_2(y)) dy + \int_{\mathbb{R}} \mathcal{F}_I\{g\}(-y) (\mathbf{i}h_1(y) + \mathbf{k}h_3(y)) dy.\end{aligned}\tag{16}$$

This is the desired result. \square

Due to the noncommutativity of quaternions we can derive the other form of equation (15) as follows.

$$\begin{aligned}
 \int_{\mathbb{R}} g(\xi) \mathcal{F}_I\{h\}(\xi) d\xi &= \int_{\mathbb{R}} g(\xi) \int_{\mathbb{R}} h(y) e^{-j2\pi\omega y} dy d\xi \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}} g(\xi) (h_0(y) + \mathbf{i}h_1(y) + \mathbf{j}h_2(y) + \mathbf{k}h_3(y)) e^{-j2\pi\omega y} dy d\xi \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}} g(\xi) h_0(y) e^{-j2\pi\omega y} dy d\xi + \int_{\mathbb{R}} \int_{\mathbb{R}} g(\xi) h_1(y) e^{j2\pi\omega y} dy d\xi \mathbf{i} \\
 &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} g(\xi) h_2(y) e^{-j2\pi\omega y} dy d\xi \mathbf{j} + \int_{\mathbb{R}} \int_{\mathbb{R}} g(\xi) h_3(y) e^{j2\pi\omega y} dy d\xi \mathbf{k} \\
 &= \int_{\mathbb{R}} \mathcal{F}_I\{g\}(y) h_0(y) dy + \int_{\mathbb{R}} \mathcal{F}_I\{g\}(-y) h_1(y) dy \mathbf{i} \\
 &\quad + \int_{\mathbb{R}} \mathcal{F}_I\{g\}(y) h_2(y) dy \mathbf{j} + \int_{\mathbb{R}} \mathcal{F}_I\{g\}(-y) h_3(y) dy \mathbf{k}. \tag{17}
 \end{aligned}$$

4. Convolution Theorem in qFT Domain

In what follows, we present convolution theorem associated with the qFT. Since the multiplication of quaternion is not commutative we obtain two types of the convolution theorems. Let us introduce the convolution definition in the qFT domain.

Definition 4.1. Given two quaternion functions $g, h \in L^2(\mathbb{R}; \mathbb{H})$, the convolution of g and h is defined by

$$(g \star h)(y) = \int_{\mathbb{R}} g(t) h(y-t) dt. \tag{18}$$

Now we consider the qFT of convolution, which describes how the convolution of two quaternion functions interacts with its qFT.

Theorem 4.1. For g, h belong to $L^2(\mathbb{R}; \mathbb{H})$. Then, the qFT of the convolution of g and h are given by

$$\mathcal{F}_I\{g \star h\}(\omega) = \mathcal{F}_I\{g\}(\omega) (\mathcal{F}_I\{h_0 + \mathbf{j}h_2\}(\omega)) + \mathcal{F}_I\{g\}(-\omega) (\mathcal{F}_I\{\mathbf{i}h_1 + \mathbf{k}h_3\}(\omega)).$$

Proof. In view of (18), we see that

$$\begin{aligned}
 \mathcal{F}_I\{g \star h\}(\omega) &= \int_{\mathbb{R}} (g \star h)(y) e^{-j2\pi\omega y} dy \\
 &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} g(t) h(x-t) e^{-j2\pi\omega y} dy \right] dt.
 \end{aligned}$$

Setting $y = x - t$ yields

$$\begin{aligned}
 & \mathcal{F}_I\{g \star h\}(\omega) \\
 &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} g(t)h(y)e^{-j2\pi\omega(t+y)} dy \right] dt \\
 &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} g(t)(h_0(y) + \mathbf{i}h_1(y) + \mathbf{j}h_2(y) + \mathbf{k}h_3(y))e^{-j2\pi\omega(t+y)} dy \right] dt \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}} g(t)(h_0(y) + \mathbf{j}h_2(y))e^{-j2\pi\omega(t+y)} dy dt + \int_{\mathbb{R}} \int_{\mathbb{R}} g(t)(\mathbf{i}h_1(y) + \mathbf{k}h_3(y))e^{-j2\pi\omega(t+y)} dy dt \\
 &= \int_{\mathbb{R}} g(t)e^{-j2\pi\omega t} dt \int_{\mathbb{R}} (h_0(y) + \mathbf{j}h_2(y))e^{-j2\pi\omega y} dy \\
 &\quad + \int_{\mathbb{R}} g(t)e^{j2\pi\omega t} dt \int_{\mathbb{R}} (\mathbf{i}h_1(y) + \mathbf{k}h_3(y))e^{-j2\pi\omega y} dy \\
 &= \mathcal{F}_I\{g\}(\omega)(\mathcal{F}_I\{h_0 + \mathbf{j}h_2\}(\omega)) + \mathcal{F}_I\{g\}(-\omega)(\mathcal{F}_I\{\mathbf{i}h_1 + \mathbf{k}h_3\}(\omega)). \tag{19}
 \end{aligned}$$

This proves the proof of the theorem. \square

The other form of the convolution theorem for the qFT may be written in following statement.

Theorem 4.2. Under the same conditions as in Theorem 4.2, one can get

$$\begin{aligned}
 \mathcal{F}_I\{g \star h\}(\omega) &= \mathcal{F}_I\{h\}(\omega)\mathcal{F}_I\{g_0\}(\omega) + \mathbf{i}\mathcal{F}_I\{h\}(\omega)\mathcal{F}_I\{g_1\}(\omega) \\
 &\quad + \mathbf{j}\mathcal{F}_I\{h\}(\omega)\mathcal{F}_I\{g_2\}(\omega) + \mathbf{k}\mathcal{F}_I\{h\}(\omega)\mathcal{F}_I\{g_3\}(\omega).
 \end{aligned}$$

Proof. Proceed equation (20) we obtain

$$\begin{aligned}
 & \mathcal{F}_I\{g \star h\}(\omega) \\
 &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} g(t)h(y)e^{-j2\pi\omega(t+y)} dy \right] dt \\
 &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} (g_0(t) + \mathbf{i}g_1(t) + \mathbf{j}g_2(t) + \mathbf{k}g_3(t))h(y)e^{-j2\pi\omega(t+y)} dy \right] dt \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}} (h(y)g_0(t) + \mathbf{i}h(y)g_1(t) + \mathbf{j}h(y)g_2(t) + \mathbf{k}h(y)g_3(t))e^{-j2\pi\omega(t+y)} dy dt \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}} h(y)g_0(t)e^{-j2\pi\omega t}e^{-j2\pi\omega t} dy dt + \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{i}h(y)g_1(t)e^{-j2\pi\omega y}e^{-j2\pi\omega t} dy dt \\
 &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{j}h(y)g_2(t)e^{-j2\pi\omega y}e^{-j2\pi\omega t} dy dt + \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{k}h(y)g_3(t)e^{-j2\pi\omega y}e^{-j2\pi\omega t} dy dt. \tag{20}
 \end{aligned}$$

In consequence,

$$\begin{aligned}
 \mathcal{F}_I\{g \star h\}(\omega) &= \mathcal{F}_I\{h\}(\omega)\mathcal{F}_I\{g_0\}(\omega) + \mathbf{i}\mathcal{F}_I\{h\}(\omega)\mathcal{F}_I\{g_1\}(\omega) \\
 &\quad + \mathbf{j}\mathcal{F}_I\{h\}(\omega)\mathcal{F}_I\{g_2\}(\omega) + \mathbf{k}\mathcal{F}_I\{h\}(\omega)\mathcal{F}_I\{g_3\}(\omega). \tag{21}
 \end{aligned}$$

The assertion is proved. \square

5. Conclusion

We have presented one-dimensional quaternion Fourier transform. We have established its properties which are generalizations of the Fourier transform. We finally demonstrated convolution theorem related one-dimensional quaternion Fourier transform.

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